Efficient Implementations of Minimum Cost Flow Algorithms

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The Minimum Cost Flow Problem

- The MCF problem is one of the most fundamental models in network flow theory.
- It is to find a *feasible flow* of a given value (k) with *minimum total cost* from a source node (s) to a target node (t) in a network with capacity constraints and arc costs.
- In most cases all data are integral and we search for an integral flow.
- Applications:
  - network design,
  - VPN allocation,
  - traffic engineering,
  - transportation,
  - resource planning, etc.

Implementation and Testing

- We implemented *six different algorithms* and many variants for the MCF problem.
- We used the **LEMON** C++ library, http://lemon.cs.elte.hu.
- Our codes are part of the library now.
- Validity and benchmark tests were performed on various large scale random networks (up to *1 million nodes* and *30 million arcs*).
- The test graphs were generated with NETGEN.

Cycle Canceling

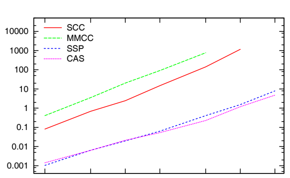
- Cycle Canceling (CC) is the simplest solution method for MCF.
- It is a *primal* method:
  - find a feasible solution;
  - at each step find a directed cycle with negative cost in the residual network and augment flow on it to saturate an arc.
- We implemented two CC algorithms:
  - a **simple cycle-canceling** (SCC) algorithm that uses Bellman-Ford algorithm for finding negative cycles and runs in $O(nm^2CU)$ time;
  - the **minimum mean cycle-canceling** (MMCC) algorithm, which runs in strongly polynomial $O(n^2m^3 \log n)$ time.

Dual Algorithms

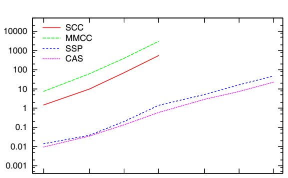
- We implemented two efficient *dual* algorithms:
  - the **successive shortest path** (SSP) algorithm,
  - the **capacity scaling** (CAS) algorithm.
- The SSP algorithm maintains an optimal flow of value $k' \leq k$ and node potentials.
- At each step it sends flow from s to t along a shortest path in the residual network with respect to reduced costs.
- It increases the flow value until the solution becomes feasible.
- It performs $O(nU)$ iterations, so it runs in $O(nU\,\mathrm{SP}(n,m,nC))$ time.
- The CAS algorithm is a more efficient variant of SSP that uses capacity scaling and runs in $O(m \log U\,\mathrm{SP}(n,m,nC))$ time.

Comparison

- The following charts compare the primal and dual algorithms.
- *deg* means the average out-degree of the nodes.



Comparison of primal and dual algorithms (deg = 5)



Comparison of primal and dual algorithms (deg = 50)

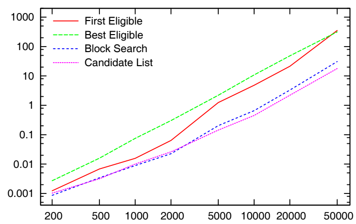
- The dual algorithms (SSP, CAS) proved to be much more efficient.

Cost Scaling

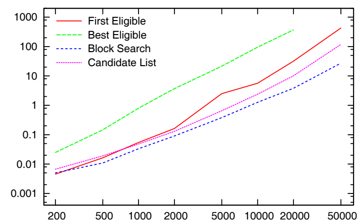
- The **cost scaling** (COS) algorithm is a generalization of the *preflow-push* algorithm by Goldberg and Tarjan.
- At each iteration it performs local *push* and *relabel* operations.
- It successively achieves an $\varepsilon/2$-optimal flow from an ε-optimal flow.
- We implemented this algorithm with various efficient heuristics that highly affect its real-time performance.

Network Simplex

- The **network simplex** (NS) algorithm is a specialized variant of the simplex method.
- In the base solutions the arcs with non-trivial flow value form a *spanning tree*.
- The crucial points of the implementation are the *data structure* used for representing spanning trees with associated data and the *pivot rules*.
- The following charts compare the four pivot rules we implemented.



Comparison of NS pivot rules (deg = 5)

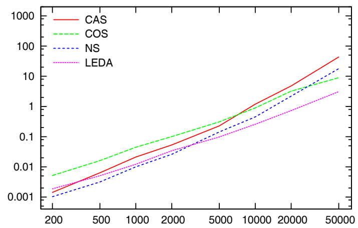


Comparison of NS pivot rules (deg = 50)

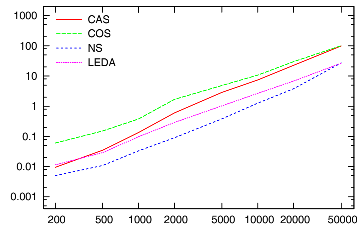
- We combined the two fastest implementations: for rather sparse graphs *Candidate List* rule is used, otherwise *Block Search* rule is used.

Comparison

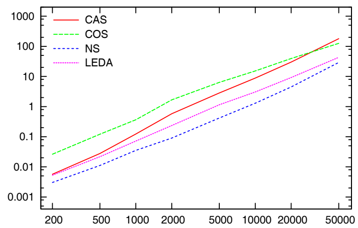
- We compare our fastest implementations and the MCF method of LEDA.
- The charts show running times in seconds as a function of the number of nodes (n).



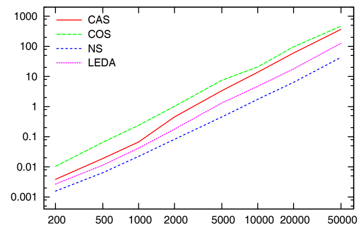
Comparison of the fastest algorithms (deg = 5)



Comparison of the fastest algorithms (deg = 50)



Comparison of the fastest algorithms (deg = 5*log(n))



Comparison of the fastest algorithms (deg = sqrt(n))

Conclusions

- There is no absolute winner.
- In most cases, especially on dense graphs NS proved to be the fastest among our implementations.
- On rather large and sparse graphs COS is the most efficient.
- CAS is usually fast, especially if the the arc capacities are relatively small.
- NS proved to be even more efficient than LEDA when
  - the network is small (it has at most 2000-5000 nodes) or
  - the network is not too sparse.
- However on sparse graphs LEDA is usually faster than all of our implementations.